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*Menger*概率度量空间压缩映象的公共不动点定理

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摘　要:在*Menger*概率度量空间中,研究一种新的压缩条件,在这个条件下得到一个新的弱相容映象对的耦合重合点和公共耦合不动点定理。定理的证明通过以下三步展开:第一步:构造并证明gx_n和gy_n是柯西序列;第二步:证明(x^*, y^*)是弱相容映象对g和T的公共耦合重合点;第三步:证明此重合点的唯一性,进而得到了g和T的公共耦合不动点也是唯一的。该定理在一定程度上推广和发展了原有结果。

关键词:*Menger*概率度量空间;耦合重合点;公共耦合不动点

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1906年,Frechet首次提出了概率度量空间的概念并对其进行了研究。近年来,*Menger*概率度量空间的理论与应用也被许多国内外学者研究,如Schweizer[1],Hadžić[2-3],Zhang等人[4-5],Chauhan等人[6],Wu等人[7-8],张树义等人[9-10]。文章引用了一些基本概念,在*Menger*概率度量空间中研究一个新的不动点定理。

1　预备知识

定义1[11]　若映象$f: R \to R$是不减的,左连续的,$\inf_{t \in R} f(t) = 0$,$\sup_{t \in R} f(t) = 1$,则映象f称为分布函数。

设D为全体分布函数,若满足

$$H(t) = \begin{cases} 1, t > 0 \\ 0, t \leqslant 0 \end{cases}$$

称H为特殊的分布函数。

定义2[12]　若映象$\Delta: [0,1] \times [0,1] \to [0,1]$满足以下条件:

(1)$\forall a \in [0,1], \Delta(a,1) = a, \Delta(0,0) = 0$;

(2)$\forall a,b \in [0,1], \Delta(a,b) = \Delta(b,a)$;

(3)$\forall a,b,c,d \in [0,1]$,若$c \geqslant a, d \geqslant b$,有$\Delta(c,d) \geqslant \Delta(a,b)$;

(4)$\forall a,b,c \in [0,1], \Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$.

则映象Δ称为三角范数(简称:t范数)。

t范数的三个典型的例子如下

(1)$\Delta_1(a,b) = \max\{a + b - 1, 0\}$;

(2)$\Delta_2(a,b) = a \cdot b$;

(3)$\Delta_3(a,b) = \min\{a,b\}$.

则满足$\Delta_1 \leqslant \Delta_2 \leqslant \Delta_3$.

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注1 假设Δ是t范数,满足$\Delta(t,t) \geqslant t, \forall t \in [0,1]$,则$\Delta \geqslant \Delta_3 \geqslant \Delta_2 \geqslant \Delta_1$.

定义3[3] 假设三元组(X, F, Δ),其中X是非空集,Δ是连续t范数,$F:X \times X \to D$,且$F_{x,y}$满足以下条件:

(1)$F_{x,y}(t) = H(t), t \in R$,当且仅当$x = y$;

(2)$F_{x,y}(t) = F_{y,x}(t), \forall x,y \in X, \forall t \in R$;

(3)$F_{x,y}(t+s) \geqslant \Delta(F_{x,z}(t), F_{z,y}(s)), \forall x,y,z \in X, \forall s,t \in R^+$.

则三元组(X, F, Δ)称为*Menger*概率度量空间(简称:*Menger PM*空间)。

定义4[13] 设三元组(X, F, Δ)为*Menger PM*空间,Δ是连续t范数,则

(1)$\{x_n\}$收敛于x^*当且仅当对$\forall \varepsilon > 0, \lambda \in (0,1)$,存在正整数$N = N(\varepsilon, \lambda)$,使得$F_{x_n,x^*}(\varepsilon) > 1 - \lambda, n > N$,$\lim\limits_{n \to \infty} F_{x_n,x^*} = 1, t > 0$;

(2)$\{x_n\}$称为柯西序列当且仅当对$\forall \varepsilon > 0, \lambda \in (0,1)$,存在正整数$N = N(\varepsilon, \lambda)$,使得$F_{x_n,x_m}(\varepsilon) > 1 - \lambda, \forall m,n \geqslant N$;

(3)(X, F, Δ)称为完备的,如果每一个柯西序列都收敛到$t, t \in X$.

定义5[5] 设(X, F, Δ)为*Menger PM*空间,A为(X, F, Δ)空间中的非空子集,如果$\sup\limits_{t>0} \inf\limits_{x,y \in A} F_{x,y}(t) = 1$,$\sup\limits_{t>0} \sup\limits_{s>t} \inf\limits_{x,y \in A} F_{x,y}(s) = 1$,则称$A$为概率有界的。

引理1[5] 设(X, F, Δ)为*Menger PM*空间,Δ是半连续t范数,那么对$\forall x_n \in X$,使得$\lim\limits_{n \to \infty} x_n = x$,且满足对$\forall y \in X, t > 0$,有$\liminf\limits_{n \to \infty} F_{x_n,y}(t) = F_{x,y}(t)$.

定义6[14] 设$(x,y) \in X \times X$,$T:X \times X \to X$,如果$T(x,y) = x, T(y,x) = y$,则(x,y)称为耦合不动点。

定义7[14] 设$(x,y) \in X \times X$,$T:X \times X \to X$,$g:X \to X$,如果$T(x,y) = gx, T(y,x) = gy$,则(x,y)称为耦合重合点。

定义8[14] 设$(x,y) \in X \times X$,$T:X \times X \to X$,$g:X \to X$,如果$T(x,y) = gx = x$,$T(y,x) = gy = y$,则(x,y)称为公共耦合不动点。

定义9[14] 设X为非空集,$T:X \times X \to X$,$g:X \to X$,如果$T(x,x) = gx = x$,则x称为耦合公共不动点。

定义10[15] 设X为非空集,$T:X \times X \to X$和$g:X \to X$是弱相容的,如果$T(x,y) = gx$,$T(y,x) = gy$,则$gT(x,y) = T(gx,gy)$.

2 主要结果

定理1 设(X, F, Δ)是*Menger PM*空间,Δ是连续的t范数,$\Delta(v,v) \geqslant v, \forall v \in [0,1]$. 设$g$、$T$为$X$中给定的两个函数,$g:X \to X$,$T:X \times X \to X$,且$g,T$是弱相容的,若满足以下条件:

(1)$T(X \times X) \subseteq g(X)$;

(2)$g(X)$是完备且概率有界的;

(3)$\forall x,y,u,v \in X, t > 0, k \in (0,1)$.

有
$$F_{T(x,y),T(u,v)}(t) \geqslant M(x,y,u,v) \tag{1}$$
其中
$$M(x,y,u,v) = \min\left\{\begin{array}{l} F_{gx,gu}\left(\frac{t}{k}\right), F_{gy,gv}\left(\frac{t}{k}\right), F_{gx,T(x,y)}\left(\frac{t}{k}\right), \\ F_{gy,T(y,x)}\left(\frac{t}{k}\right), F_{gu,T(u,v)}\left(\frac{2t}{k}\right), F_{gv,T(v,u)}\left(\frac{2t}{k}\right), \\ F_{gx,T(u,v)}\left(\frac{t}{k}\right), F_{gy,T(v,u)}\left(\frac{t}{k}\right), F_{gu,T(x,y)}\left(\frac{2t}{k}\right), F_{gv,T(y,x)}\left(\frac{2t}{k}\right) \end{array}\right\}$$
则g和T有唯一的公共耦合不动点。

证明 因为$T(X \times X) \subseteq g(X)$,假设$(x_0,y_0) \in X \times X$,存在$(x_1,y_1) \in X \times X$,使得$gx_1 = T(x_0,y_0)$,

$gy_1 = T(y_0,x_0)$;存在$(x_2,y_2) \in X \times X$,使得$gx_2 = T(x_1,y_1)$,$gy_2 = T(y_1,x_1)$;以此类推,就能得到两个点列$\{x_n\}$和$\{y_n\}$.

$$gx_{n+1} = T(x_n,y_n),gy_n = T(y_n,x_n),n = 0,1,2,\cdots \tag{2}$$

将$(x,y) = (x_n,y_n)$和$(u,v) = (x_{n+i},y_{n+i})$代入式(1),有

$$F_{T(x_n,y_n),T(x_{n+i},y_{n+i})}(t) \geqslant M(x_n,y_n,x_{n+i},y_{n+i}) \tag{3}$$

其中

$$\begin{aligned} &M(x_n,y_n,x_{n+i},y_{n+i}) \\ &= \min\left\{\begin{aligned} &\mathrm{F}_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i}}\left(\frac{t}{k}\right),F_{gx_n,T(x_n,y_n)}\left(\frac{t}{k}\right),\\ &F_{gy_n,T(x_n,y_n)}\left(\frac{t}{k}\right),F_{gx_{n+i},T(x_{n+i},y_{n+i})}\left(\frac{2t}{k}\right),F_{gy_{n+i},T(y_{n+i},x_{n+i})}\left(\frac{2t}{k}\right),\\ &F_{gx_n,T(x_{n+i},y_{n+i})}\left(\frac{t}{k}\right),F_{gy_n,T(y_{n+i},x_{n+i})}\left(\frac{t}{k}\right),F_{gx_{n+i},T(x_n,y_n)}\left(\frac{2t}{k}\right),F_{gy_{n+i},T(y_n,x_n)}\left(\frac{2t}{k}\right) \end{aligned}\right\} \\ &= \min\left\{\begin{aligned} &F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),\\ &F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right),F_{gx_{n+i},gx_{n+i+1}}\left(\frac{2t}{k}\right),F_{gy_{n+i},gy_{n+i+1}}\left(\frac{2t}{k}\right),\\ &F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i+1}}\left(\frac{t}{k}\right),F_{gx_{n+i},gx_{n+1}}\left(\frac{2t}{k}\right),F_{gy_{n+i},gy_{n+1}}\left(\frac{2t}{k}\right) \end{aligned}\right\} \end{aligned} \tag{4}$$

由定义3和注1,可得

$$F_{gx_{n+i},gx_{n+1}}\left(\frac{2t}{k}\right) \geqslant \Delta\left(F_{gx_{n+i},gx_n}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right)\right) \geqslant \min\left\{F_{gx_{n+i},gx_n}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right)\right\}$$

所以

$$F_{gx_{n+i},gx_{n+1}}\left(\frac{2t}{k}\right) \geqslant \min\left\{F_{gx_{n+i},gx_n}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right)\right\} \tag{5}$$

同理可得

$$F_{gy_{n+i},gy_{n+1}}\left(\frac{2t}{k}\right) \geqslant \min\left\{F_{gy_{n+i},gy_n}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right)\right\} \tag{6}$$

由定义3和注1,可得

$$F_{gx_{n+i},gx_{n+i+1}}\left(\frac{2t}{k}\right) \geqslant \Delta\left(F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right)\right) \geqslant \min\left\{F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right)\right\}$$

所以

$$F_{gx_{n+i},gx_{n+i+1}}\left(\frac{2t}{k}\right) \geqslant \min\left\{F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right)\right\} \tag{7}$$

同理可得

$$F_{gy_{n+i},gy_{n+i+1}}\left(\frac{2t}{k}\right) \geqslant \min\left\{F_{gy_n,gy_{n+i}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i+1}}\left(\frac{t}{k}\right)\right\} \tag{8}$$

将式(5)~(8)代入式(4),可得

$$M(x_n,y_n,x_{n+i},y_{n+i}) = \min\left\{\begin{aligned} &F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right),\\ &F_{gy_n,gy_{n+i}}\left(\frac{t}{k}\right)F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i+1}}\left(\frac{t}{k}\right) \end{aligned}\right\} \tag{9}$$

将式(9)代入式(3),可以得到

$$F_{gx_{n+1},gx_{n+i+1}}(t) \geqslant M(x_n,y_n,x_{n+i},y_{n+i}) = \min\left\{\begin{aligned} &F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right),\\ &F_{gy_n,gy_{n+i}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i+1}}\left(\frac{t}{k}\right) \end{aligned}\right\} \tag{10}$$

同理可得

$$F_{gy_{n+1},gy_{n+i+1}}(t) \geqslant M(y_n,x_n,y_{n+i},x_{n+i}) = \min\left\{\begin{array}{l} F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right), \\ F_{gy_n,gy_{n+i}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i+1}}\left(\frac{t}{k}\right) \end{array}\right\} \tag{11}$$

结合式(10)和式(11),可得

$$\min\{F_{gx_{n+1},gx_{n+i+1}}(t),F_{gy_{n+1},gy_{n+i+1}}(t)\} \geqslant \min\left\{\begin{array}{l} F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right), \\ F_{gy_n,gy_{n+i}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i+1}}\left(\frac{t}{k}\right) \end{array}\right\} \tag{12}$$

同理,可以得到以下结果

$$\min\left\{F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i+1}}\left(\frac{t}{k}\right)\right\} \geqslant \min\left\{\begin{array}{l} F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+i}}\left(\frac{t}{k^2}\right), \\ F_{gy_{n-1},gy_{n+i}}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+i+1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+i+1}}\left(\frac{t}{k^2}\right) \end{array}\right\} \tag{13}$$

$$\min\left\{F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right)\right\} \geqslant \min\left\{\begin{array}{l} F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+i-1}}\left(\frac{t}{k^2}\right), \\ F_{gy_{n-1},gy_{n+i-1}}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+i}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+i}}\left(\frac{t}{k^2}\right) \end{array}\right\} \tag{14}$$

将$(x,y)=(x_n,y_n)$和$(u,v)=(x_{n+1},y_{n+1})$代入式(1),有

$$F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right) = F_{T(x_{n-1},y_{n-1}),T(x_n,y_n)}\left(\frac{t}{k}\right) \geqslant M(x_{n-1},y_{n-1},x_n,y_n) \tag{15}$$

其中

$$M(x_{n-1},y_{n-1},x_n,y_n) = \min\left\{\begin{array}{l} F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},T(x_{n-1},y_{n-1})}\left(\frac{t}{k^2}\right), \\ F_{gy_{n-1},T(y_{n-1},x_{n-1})}\left(\frac{t}{k^2}\right),F_{gx_n,T(x_n,y_n)}\left(\frac{2t}{k^2}\right),F_{gy_n,T(y_n,x_n)}\left(\frac{2t}{k^2}\right), \\ F_{gx_{n-1},T(x_n,y_n)}\left(\frac{t}{k^2}\right),F_{gy_{n-1},T(y_n,x_n)}\left(\frac{t}{k^2}\right),F_{gx_n,T(x_{n-1},y_{n-1})}\left(\frac{2t}{k^2}\right),F_{gy_n,T(y_{n-1},x_{n-1})}\left(\frac{2t}{k^2}\right) \end{array}\right\}$$

$$= \min\left\{\begin{array}{l} F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right), \\ F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_n,gx_{n+1}}\left(\frac{2t}{k^2}\right),F_{gy_n,gy_{n+1}}\left(\frac{2t}{k^2}\right), \\ F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+1}}\left(\frac{t}{k^2}\right),F_{gx_n,gx_n}\left(\frac{2t}{k^2}\right),F_{gy_n,gy_n}\left(\frac{2t}{k^2}\right) \end{array}\right\}$$

$$= \min\left\{\begin{array}{l} F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_n,gx_{n+1}}\left(\frac{2t}{k^2}\right) \\ F_{gy_n,gy_{n+1}}\left(\frac{2t}{k^2}\right),F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+1}}\left(\frac{t}{k^2}\right) \end{array}\right\} \tag{16}$$

由定义3和注1,可得

$$F_{gx_n,gx_{n+1}}\left(\frac{2t}{k^2}\right) \geqslant \Delta\left(F_{gx_{n-i},gx_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right)\right) \geqslant \min\left\{F_{gx_{n-i},gx_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right)\right\}$$

所以

$$F_{gx_n,gx_{n+1}}\left(\frac{2t}{k^2}\right) \geqslant \min\left\{F_{gx_{n-i},gx_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right)\right\} \tag{17}$$

同理可得

$$F_{gy_n,gy_{n+1}}\left(\frac{2t}{k^2}\right) \geqslant \min\left\{F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+1}}\left(\frac{t}{k^2}\right)\right\} \tag{18}$$

将式(17)和式(18)代入式(16),可得

$$M(x_n,y_n,x_{n+1},y_{n+1}) = \min\left\{F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+1}}\left(\frac{t}{k^2}\right)\right\} \tag{19}$$

由式(15)和式(19)可知

$$F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right) \geqslant M(x_{n-1},y_{n-1},x_n,y_n) = \min\left\{F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+1}}\left(\frac{t}{k^2}\right)\right\} \tag{20}$$

同理可得

$$F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right) \geqslant \min\left\{F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+1}}\left(\frac{t}{k^2}\right)\right\} \tag{21}$$

由式(20)和式(21)可知

$$\min\left\{F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right)\right\} \geqslant \min\left\{F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+1}}\left(\frac{t}{k^2}\right)\right\} \tag{22}$$

将式(13)、式(14)和式(22)代入式(12)可得

$$\begin{aligned}
\min\{F_{gx_{n+1},gx_{n+i+1}}(t),F_{gy_{n+1},gy_{n+i+1}}(t)\} &\geqslant \min\left\{\begin{aligned}&F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right),\\&F_{gy_n,gy_{n+i}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i+1}}\left(\frac{t}{k}\right)\end{aligned}\right\}\\
&\geqslant \min\left\{\begin{aligned}&F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right),\\&F_{gy_{n-1},gy_{n+1}}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+i-1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+i-1}}\left(\frac{t}{k^2}\right),\\&F_{gx_{n-1},gx_{n+i}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+i}}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+i+1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+i+1}}\left(\frac{t}{k^2}\right)\end{aligned}\right\}
\end{aligned}$$

有归纳分析出的规律

$$\begin{aligned}
\min\{F_{gx_{n+1},gx_{n+i+1}}(t),F_{gy_{n+1},gy_{n+i+1}}(t)\} &\geqslant \min\left\{\begin{aligned}&F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i}}\left(\frac{t}{k}\right),\\&F_{gy_n,gy_{n+i}}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+i+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+i+1}}\left(\frac{t}{k}\right)\end{aligned}\right\}\\
&\geqslant \min\left\{\begin{aligned}&F_{gx_{n-1},gx_n}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_n}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+1}}\left(\frac{t}{k^2}\right),\\&F_{gy_{n-1},gy_{n+1}}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+i-1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+i-1}}\left(\frac{t}{k^2}\right),\\&F_{gx_{n-1},gx_{n+i}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+i}}\left(\frac{t}{k^2}\right),F_{gx_{n-1},gx_{n+i+1}}\left(\frac{t}{k^2}\right),F_{gy_{n-1},gy_{n+i+1}}\left(\frac{t}{k^2}\right)\end{aligned}\right\}\\
&\geqslant \min\left\{\begin{aligned}&F_{gx_{n-2},gx_{n-1}}\left(\frac{t}{k^3}\right),F_{gy_{n-2},gy_{n-1}}\left(\frac{t}{k^3}\right),F_{gx_{n-2},gx_n}\left(\frac{t}{k^3}\right),\\&F_{gy_{n-2},gy_n}\left(\frac{t}{k^3}\right),F_{gx_{n-2},gx_{n+1}}\left(\frac{t}{k^3}\right),F_{gy_{n-2},gy_{n+1}}\left(\frac{t}{k^3}\right),\\&F_{gx_{n-2},gx_{n+i-2}}\left(\frac{t}{k^3}\right),F_{gy_{n-2},gy_{n+i-2}}\left(\frac{t}{k^3}\right),F_{gx_{n-2},gx_{n+i-1}}\left(\frac{t}{k^3}\right),F_{gy_{n-2},gy_{n+i-1}}\left(\frac{t}{k^3}\right),\\&F_{gx_{n-2},gx_{n+i}}\left(\frac{t}{k^3}\right),F_{gy_{n-2},gy_{n+i}}\left(\frac{t}{k^3}\right),F_{gx_{n-2},gx_{n+i+1}}\left(\frac{t}{k^3}\right),F_{gy_{n-2},gy_{n+i+1}}\left(\frac{t}{k^3}\right)\end{aligned}\right\}
\end{aligned}$$

$$\geqslant \cdots \geqslant \min\left\{\begin{array}{l} F_{gx_1,gx_2}\left(\frac{t}{k^n}\right),F_{gy_1,gy_2}\left(\frac{t}{k^n}\right),F_{gx_1,gx_3}\left(\frac{t}{k^n}\right),F_{gy_1,gy_3}\left(\frac{t}{k^n}\right),\cdots,F_{gx_1,gx_{n+1}}\left(\frac{t}{k^n}\right),\\ \cdots,F_{gx_1,gx_{i+n}}\left(\frac{t}{k^n}\right),F_{gy_1,gy_{i+n}}\left(\frac{t}{k^n}\right),F_{gx_1,gx_{i+n+1}}\left(\frac{t}{k^n}\right),F_{gy_1,gy_{i+n+1}}\left(\frac{t}{k^n}\right),\\ F_{gy_1,gy_{n+1}}\left(\frac{t}{k^n}\right),F_{gx_1,gx_{i+1}}\left(\frac{t}{k^n}\right),F_{gy_1,gy_{i+1}}\left(\frac{t}{k^n}\right),F_{gx_1,gx_{i+2}}\left(\frac{t}{k^n}\right),F_{gy_1,gy_{i+2}}\left(\frac{t}{k^n}\right),\\ F_{gx_{n-2},gx_{n+i}}\left(\frac{t}{k^3}\right),F_{gy_{n-2},gy_{n+i}}\left(\frac{t}{k^3}\right),F_{gx_{n-2},gx_{n+i+1}}\left(\frac{t}{k^3}\right),F_{gy_{n-2},gy_{n+i+1}}\left(\frac{t}{k^3}\right) \end{array}\right\}$$

$$\geqslant \min\left\{\inf_{q\in\{gx_j\}_{j=2}^{n+i+1}} F_{gx_1,gq}\left(\frac{t}{k^n}\right),\inf_{p\in\{gy_j\}_{j=2}^{n+i+1}} F_{gx_1,gp}\left(\frac{t}{k^n}\right)\right\}$$

$$\geqslant \min\left\{\sup_{u<\frac{t}{k^n}}\inf_{q\in\{gx_j\}_{j=2}^{n+i+1}} F_{gx_1,gq}(u),\sup_{u<\frac{t}{k^n}}\inf_{p\in\{gy_j\}_{j=2}^{n+i+1}} F_{gx_1,gp}(u)\right\}$$

因为$g(X)$是概率有界的,当$n\to\infty$时,有

$$\lim_{n\to\infty}\min\{F_{gx_{n+1},gx_{n+i+1}}(t),F_{gy_{n+1},gy_{n+i+1}}(t)\}\geqslant \min\left\{\lim_{n\to\infty}\sup_{u<\frac{t}{k^n}}\inf_{q\in\{gx_j\}_{j=2}^{n+i+1}} F_{gx_1,gq}(u),\lim_{n\to\infty}\sup_{u<\frac{t}{k^n}}\inf_{p\in\{gy_j\}_{j=2}^{n+i+1}} F_{gx_1,gp}(u)\right\}$$

$$=\begin{cases}\min\left\{\sup_{u<\frac{t}{k^n}}\inf_{q\in\{gx_j\}_{j=2}^{n+i+1}} F_{gx_1,gq}(u),\sup_{u<\frac{t}{k^n}}\inf_{p\in\{gy_j\}_{j=2}^{n+i+1}} F_{gx_1,gp}(u)\right\}, & t>0;\\ 0,t=0\end{cases}$$

$$=\begin{cases}1,t>0\\ 0,t=0\end{cases}$$

所以

$$\lim_{n\to\infty}\min\{F_{gx_{n+1},gx_{n+i+1}}(t),F_{gy_{n+1},gy_{n+i+1}}(t)\}=1,\forall t>0$$

以下证明gx_n和gy_n是柯西序列。因为$g(X)$是完备的,假设

$$gx_n\to gx^*,gy_n\to gy^*(n\to\infty) \tag{23}$$

其中$(x^*,y^*)\in X\times X$. 将$(x,y)=(x_n,y_n)$和$(u,v)=(x^*,y^*)$代入式(1),有

$$F_{gx_{n+1},T(x^*,y^*)}(t)=F_{T(x_n,y_n),T(x^*,y^*)}(t)\geqslant M(x_n,y_n,x^*,y^*) \tag{24}$$

其中

$$M(x_n,y_n,x^*,y^*)=\min\left\{\begin{array}{l} F_{gx_n,gx^*}\left(\frac{t}{k}\right),F_{gy_n,gy^*}\left(\frac{t}{k}\right),F_{gx_n,T(x_n,y_n)}\left(\frac{t}{k}\right),F_{gy_n,T(y_n,x_n)}\left(\frac{t}{k}\right),F_{gx^*,T(x^*,y^*)}\left(\frac{2t}{k}\right)\\ F_{gy^*,T(y^*,x^*)}\left(\frac{2t}{k}\right),F_{gx_n,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy_n,T(y^*,x^*)}\left(\frac{t}{k}\right),F_{gx^*,T(x_n,y_n)}\left(\frac{2t}{k}\right),F_{gy^*,T(y_n,x_n)}\left(\frac{2t}{k}\right) \end{array}\right\}$$

$$=\min\left\{\begin{array}{l} F_{gx_n,gx^*}\left(\frac{t}{k}\right),F_{gy_n,gy^*}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),\\ F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right),F_{gx^*,T(x^*,y^*)}\left(\frac{2t}{k}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{2t}{k}\right),\\ F_{gx_n,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy_n,T(y^*,x^*)}\left(\frac{t}{k}\right),F_{gx^*,gx_{n+1}}\left(\frac{2t}{k}\right),F_{gy^*,gy_{n+1}}\left(\frac{2t}{k}\right) \end{array}\right\} \tag{25}$$

由定义3和注1可得

$$F_{gx^*,T(x^*,y^*)}\left(\frac{2t}{k}\right)\geqslant\Delta\left(F_{gx^*,gx_n}\left(\frac{t}{k}\right),F_{gx_n,T(x^*,y^*)}\left(\frac{t}{k}\right)\right)\geqslant\min\left\{F_{gx^*,gx_n}\left(\frac{t}{k}\right),F_{gx_n,T(x^*,y^*)}\left(\frac{t}{k}\right)\right\}$$

所以

$$F_{gx^*,T(x^*,y^*)}\left(\frac{2t}{k}\right)\geqslant\min\left\{F_{gx^*,gx_n}\left(\frac{t}{k}\right),F_{gx_n,T(x^*,y^*)}\left(\frac{t}{k}\right)\right\} \tag{26}$$

同理可得
$$F_{gy^*,T(y^*,x^*)}\left(\frac{2t}{k}\right) \geqslant \min\left\{F_{gy^*,gy_n}\left(\frac{t}{k}\right),F_{gx_n,T(y^*,x^*)}\left(\frac{t}{k}\right)\right\} \tag{27}$$

将式(26)和式(27)代入式(25),可以得到

$$M(x^*,y^*,\ x_n,y_n) \geqslant \min\left\{\begin{array}{l}F_{gx_n,gx^*}\left(\frac{t}{k}\right),F_{gy_n,gy^*}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right)\\ F_{gx_n,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy_n,T(y^*,x^*)}\left(\frac{t}{k}\right),F_{gx^*,gx_{n+1}}\left(\frac{2t}{k}\right),F_{gy^*,gy_{n+1}}\left(\frac{2t}{k}\right)\end{array}\right\} \tag{28}$$

将式(28)代入式(24),可知

$$F_{gx_{n+1},T(x^*,y^*)}(t) \geqslant \min\left\{\begin{array}{l}F_{gx_n,gx^*}\left(\frac{t}{k}\right),F_{gy_n,gy^*}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right),\\ F_{gx_n,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy_n,T(y^*,x^*)}\left(\frac{t}{k}\right),F_{gx^*,gx_{n+1}}\left(\frac{2t}{k}\right),F_{gy^*,gy_{n+1}}\left(\frac{2t}{k}\right)\end{array}\right\} \tag{29}$$

结合式(23),在上式中,当$n\to\infty$时,有

$$F_{gx^*,T(x^*,y^*)}(t) = \lim_{n\to\infty} F_{gx_{n+1},T(x^*,y^*)}(t) \geqslant \lim_{n\to\infty}\min\left\{\begin{array}{l}F_{gx_n,gx^*}\left(\frac{t}{k}\right),F_{gy_n,gy^*}\left(\frac{t}{k}\right),F_{gx_n,gx_{n+1}}\left(\frac{t}{k}\right),F_{gy_n,gy_{n+1}}\left(\frac{t}{k}\right),\\ F_{gx_n,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy_n,T(y^*,x^*)}\left(\frac{t}{k}\right),F_{gx^*,gx_{n+1}}\left(\frac{2t}{k}\right),F_{gy^*,gy_{n+1}}\left(\frac{2t}{k}\right)\end{array}\right\}$$

$$= \min\left\{1,1,1,1,F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k}\right),1,1\right\}$$

$$= \min\left\{F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k}\right)\right\}$$

所以

$$F_{gx^*,T(x^*,y^*)}(t) \geqslant \min\left\{F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k}\right)\right\} \tag{30}$$

同理可得

$$F_{gy^*,T(y^*,x^*)}(t) \geqslant \min\left\{F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k}\right)\right\} \tag{31}$$

由式(30)和式(31)可知

$$\min\left\{F_{gx^*,T(x^*,y^*)}(t),F_{gy^*,T(y^*,x^*)}(t)\right\} \geqslant \min\left\{F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k}\right)\right\}$$

重复以上步骤,可得

$$\min\left\{F_{gx^*,T(x^*,y^*)}(t),F_{gy^*,T(y^*,x^*)}(t)\right\} \geqslant \min\left\{F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k}\right)\right\}$$

$$\geqslant \min\left\{F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k^2}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k^2}\right)\right\} \geqslant \cdots \geqslant \min\left\{F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k^m}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k^m}\right)\right\}$$

所以,有
$$\min\left\{F_{gx^*,T(x^*,y^*)}(t),F_{gy^*,T(y^*,x^*)}(t)\right\} \geqslant \min\left\{F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k^m}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k^m}\right)\right\} \tag{32}$$

当$m\to\infty$时,有

$$\min\left\{F_{gx^*,T(x^*,y^*)}(t),F_{gy^*,T(y^*,x^*)}(t)\right\} \geqslant \lim_{m\to\infty}\min\left\{F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k^m}\right),F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k^m}\right)\right\}$$

$$\geqslant \lim_{m\to\infty}\min\left\{\sup_{v<\frac{t}{k^m}} F_{gx^*,T(x^*,y^*)}(v),\ \sup_{v<\frac{t}{k^m}} F_{gy^*,T(y^*,x^*)}(v)\right\}$$

$$= \begin{cases} \min\left\{ \sup\limits_{v>0} F_{gx^*,T(x^*,y^*)}(v), \sup\limits_{v>0} F_{gy^*,T(y^*,x^*)}(v) \right\}, t > 0 \\ 0, t = 0 \end{cases}$$

$$= \begin{cases} 1, t > 0 \\ 0, t = 0 \end{cases}$$

因此,当$t > 0$时,有$F_{gx^*,T(x^*,y^*)}(t) = 1, F_{gy^*,T(y^*,x^*)}(t) = 1$. 所以可以得到$T(x^*,y^*) = gx^*$, $T(y^*,x^*) = gy^*$,即(x^*,y^*)是g和T的公共耦合重合点。接下来,将证明这个重合点的唯一性。

设$(a,b) \in X \times X$是g、T的公共耦合不动点,则$T(a,b) = ga$, $T(b,a) = gb$. 将$(x,y) = (x^*,y^*)$与$(u,v) = (a,b)$代入式(1),得到

$$F_{gx^*,ga}(t) = F_{T(x^*,y^*),T(a,b)}(t) \geqslant M(x^*,y^*,a,b) \tag{33}$$

其中

$$M(x^*,y^*,a,b) = \min\left\{ \begin{aligned} & F_{gx^*,ga}\left(\frac{t}{k}\right), F_{gy^*,gb}\left(\frac{t}{k}\right), F_{gx^*,T(x^*,y^*)}\left(\frac{t}{k}\right), F_{gy^*,T(y^*,x^*)}\left(\frac{t}{k}\right), F_{ga,T(a,b)}\left(\frac{2t}{k}\right), \\ & F_{gb,T(b,a)}\left(\frac{2t}{k}\right), F_{gx^*,T(a,b)}\left(\frac{t}{k}\right), F_{gy^*,T(b,a)}\left(\frac{t}{k}\right), F_{ga,T(x^*,y^*)}\left(\frac{2t}{k}\right), F_{gb,T(y^*,x^*)}\left(\frac{2t}{k}\right) \end{aligned} \right\}$$

$$= \min\left\{ \begin{aligned} & F_{gx^*,ga}\left(\frac{t}{k}\right), F_{gy^*,gb}\left(\frac{t}{k}\right), F_{gx^*,gx^*}\left(\frac{t}{k}\right), F_{gy^*,gy^*}\left(\frac{t}{k}\right), F_{ga,ga}\left(\frac{2t}{k}\right), \\ & F_{gb,gb}\left(\frac{2t}{k}\right), F_{gx^*,ga}\left(\frac{t}{k}\right), F_{gy^*,gb}\left(\frac{t}{k}\right), F_{ga,gx^*}\left(\frac{2t}{k}\right), F_{gb,gy^*}\left(\frac{2t}{k}\right) \end{aligned} \right\}$$

$$= \min\left\{ F_{gx^*,ga}\left(\frac{t}{k}\right), F_{gy^*,gb}\left(\frac{t}{k}\right) \right\}$$

因此,可以得到

$$M(x^*,y^*,a,b) = \min\left\{ F_{gx^*,ga}\left(\frac{t}{k}\right), F_{gy^*,gb}\left(\frac{t}{k}\right) \right\} \tag{34}$$

将式(34)代入式(33),可得

$$F_{gx^*,ga}(t) \geqslant \min\left\{ F_{gx^*,ga}\left(\frac{t}{k}\right), F_{gy^*,gb}\left(\frac{t}{k}\right) \right\} \tag{35}$$

同理可得

$$F_{gy^*,gb}(t) \geqslant \min\left\{ F_{gx^*,ga}\left(\frac{t}{k}\right), F_{gy^*,gb}\left(\frac{t}{k}\right) \right\} \tag{36}$$

结合式(35)和式(36),则

$$\min\{F_{gx^*,ga}(t), F_{gy^*,gb}(t)\} \geqslant \min\left\{ F_{gx^*,ga}\left(\frac{t}{k}\right), F_{gy^*,gb}\left(\frac{t}{k}\right) \right\}$$

以此类推,重复以上步骤,可得以下不等式

$$\min\{F_{gx^*,ga}(t), F_{gy^*,gb}(t)\} \geqslant \min\left\{ F_{gx^*,ga}\left(\frac{t}{k}\right), F_{gy^*,gb}\left(\frac{t}{k}\right) \right\}$$

$$\geqslant \min\left\{ F_{gx^*,ga}\left(\frac{t}{k^2}\right), F_{gy^*,gb}\left(\frac{t}{k^2}\right) \right\} \geqslant \cdots \geqslant \min\left\{ F_{gx^*,ga}\left(\frac{t}{k^\lambda}\right), F_{gy^*,gb}\left(\frac{t}{k^\lambda}\right) \right\}$$

则

$$\min\{F_{gx^*,ga}(t), F_{gy^*,gb}(t)\} \geqslant \min\left\{ F_{gx^*,ga}\left(\frac{t}{k^\lambda}\right), F_{gy^*,gb}\left(\frac{t}{k^\lambda}\right) \right\} \tag{37}$$

令上式中的$\lambda \to \infty$,可知

$$\min\{F_{gx^*,ga}(t), F_{gy^*,gb}(t)\} \geqslant \min\left\{ \lim_{\lambda\to\infty} F_{gx^*,ga}\left(\frac{t}{k^\lambda}\right), \lim_{\lambda\to\infty} F_{gy^*,gb}\left(\frac{t}{k^\lambda}\right) \right\}$$

$$\geqslant \min\left\{\lim_{\lambda\to\infty}\sup_{u<\frac{t}{k^{\lambda}}} F_{gx^*,ga}(u),\ \lim_{\lambda\to\infty}\sup_{u<\frac{t}{k^{\lambda}}} F_{gy^*,gb}(u)\right\} = \begin{cases}\min\left\{\sup\limits_{u>0} F_{gx^*,ga}(u),\ \sup\limits_{v>0} F_{gy^*,gb}(u)\right\}, t>0\\ 0, t=0\end{cases} = \begin{cases}1, t>0\\ 0, t=0\end{cases}$$

由上式可知,当$t>0$时,有$\min\{F_{gx^*,ga}(t),F_{gy^*,gb}(t)\}=1$. 则$gx^*=ga$,$gy^*=gb$,即可证得$g$和$T$的公共耦合重合点是唯一的。

令$gx^*=T(x^*,y^*)=m$,$gy^*=T(y^*,x^*)=n$,其中$(m,n)\in X\times X$. 因为g和T是弱相容的,则有

$$m=gm=gT(x^*,y^*)=T(gx^*,gy^*)=T(m,n), n=gn=gT(y^*,x^*)=T(gy^*,gx^*)=T(n,m)$$

由上式可知,$m=gm=T(m,n)$,$n=gn=T(n,m)$,即(gm,gn)是g和T的公共耦合重合点,(m,n)是g和T的公共耦合不动点。因为重合点是唯一的,所以g和T的公共耦合不动点也是唯一的。

推论1 设(X,F,Δ)是Menger PM空间,Δ是连续的t范数,$\Delta(v,v)\geqslant v$,$\forall v\in[0,1]$. 设g,T为X中的两个给定函数,$g:X\to X$,$T:X\times X\to X$,且g和T是弱相容的,若满足以下条件:

(1)$T(X\times X)\subseteq g(X)$;

(2)$g(X)$是完备且概率有界的;

(3)$\forall x,y,u,v\in X,t>0,k\in(0,1)$,有$F_{T(x,y),T(u,v)}(t)\geqslant M(x,y,u,v)$.

其中

$$M(x,y,u,v)=\min\left\{\begin{aligned}&F_{x,u}\left(\frac{t}{k}\right),F_{y,v}\left(\frac{t}{k}\right),F_{x,T(x,y)}\left(\frac{t}{k}\right),\\&F_{y,T(y,x)}\left(\frac{t}{k}\right),F_{u,T(u,v)}\left(\frac{2t}{k}\right),F_{v,T(v,u)}\left(\frac{2t}{k}\right),\\&F_{x,T(u,v)}\left(\frac{t}{k}\right),F_{y,T(v,u)}\left(\frac{t}{k}\right),F_{u,T(x,y)}\left(\frac{2t}{k}\right),F_{v,T(y,x)}\left(\frac{2t}{k}\right)\end{aligned}\right\}$$

则g和T有唯一的公共耦合不动点。

定理2 设(X,F,Δ)是Menger PM空间,Δ是连续的t范数,$\Delta(v,v)\geqslant v$,$\forall v\in[0,1]$. 设g,T为X中的两个给定函数,$g:X\to X$,$T:X\times X\to X$,且g,T是弱相容的,若满足以下条件:

(1)$T(X\times X)\subseteq g(X)$;

(2)$g(X)$是完备且概率有界的;

(3)$\exists\{k_i\}_{i=1}^{10}\in(0,1)$,使得$\sum_{i=1}^{10}k_i\in(0,1)$,$\forall x,y,u,v\in X,t>0,k\in(0,1)$,有$F_{T(x,y),T(u,v)}(t)\geqslant M(x,y,u,v)$.

其中

$$\begin{aligned}M(x,y,u,v)=&F_{gx,gu}\left(\frac{t}{k_1}\right)+F_{gy,gv}\left(\frac{t}{k_2}\right)+F_{gx,T(x,y)}\left(\frac{t}{k_3}\right)+F_{gy,T(y,x)}\left(\frac{t}{k_4}\right)+F_{gu,T(u,v)}\left(\frac{2t}{k_5}\right)+F_{gv,T(v,u)}\left(\frac{2t}{k_6}\right)\\&+F_{gx,T(u,v)}\left(\frac{t}{k_7}\right)+F_{gy,T(v,u)}\left(\frac{t}{k_8}\right)+F_{gu,T(x,y)}\left(\frac{2t}{k_9}\right)+F_{gv,T(y,x)}\left(\frac{2t}{k_{10}}\right)\end{aligned}$$

则g和T有唯一的公共耦合不动点。

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A Common Fixed Point Theorem of Contractive Mappingsin *Menger* Probabilistic Metric Spaces

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Abstract：In the framework of a *Menger* probabilistic metric spaces，we introduce a new class of contraction condition，and proves some new coupled coincidence point and common coupled fixed point theorems. This proof is completed through the following three steps. Step1：we show that gx_n and gy_n are Cauchy sequences；Step2：it will be proved that (x^*,y^*) is the common coupled point of coincidence of g and T；Step 3：the uniqueness of the common coupled point of coincidence of g and T is proved. This theorem improves the corresponding results in some references.

Keywords：*Menger* probabilistic metric spaces；Couple coincidence point；Common couple fixed point